

NON-LOCAL OPERATORS, NON-ARCHIMEDEAN PARABOLIC-TYPE EQUATIONS WITH VARIABLE COEFFICIENTS AND MARKOV PROCESSES

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ABSTRACT. In this article, we introduce a new class of parabolic-type pseudo differential equations with variable coefficients over the p -adics. We establish the existence and uniqueness of solutions for the Cauchy problem associated with these equations. The fundamental solutions of these equations are connected with Markov processes. Some of these equations are related to new models of complex systems.

1. INTRODUCTION

Stochastic processes on p -adic spaces, or more generally on ultrametric spaces, have been studied extensively due to its connections with models of complex systems, see e.g. [2]-[3], [4]-[6], [8], [9], [11], [13], [14], [15], [16], [22], and the references therein. In [4]-[6], Avetisov et al. introduced a new class of models for complex systems based on p -adic analysis, these models can be applied, for instance, to the study the relaxation of biological complex systems. From a mathematical point view, in these models the time-evolution of a complex system is described by a p -adic master equation (a parabolic-type pseudodifferential equation) which controls the time-evolution of a transition function of a random walk on an ultrametric space, and the random walk describes the dynamics of the system in the space of configurational states which is approximated by an ultrametric space (\mathbb{Q}_p) . The simplest type of master equation is the one-dimensional p -adic heat equation. This equation was introduced in the book of Vladimirov, Volovich and Zelenov [22, Section XVI]. In [13, Chapters 4, 5] Kochubei presented a general theory for one-dimensional parabolic-type pseudodifferential equations with variable coefficients, whose fundamental solutions are transition density functions for Markov processes in the p -adic line, see also [17], [18], [21]. In [23], the second author introduced p -adic analogs for the n -dimensional elliptic operators and studied the corresponding heat equations and the associated Markov processes, see also [7], [21].

In [8], the authors introduced a new type of non-local operators which are naturally connected with parabolic-type pseudodifferential equations. Building up on [8] and [13]-[12], in this article, we introduce a new class of parabolic-type pseudodifferential equations with variable coefficients, which contains the one-dimensional p -adic heat equation of [22], the equations studied by Kochubei in [13], and the equations studied by Rodríguez-Vega in [17]. Our theory is not applicable to the

2000 *Mathematics Subject Classification.* Primary 35K90, 60J25; Secondary 26E30.

Key words and phrases. Parabolic-type equations, diffusion, dynamics of disordered systems, Markov processes, p -adic fields, non-Archimedean analysis.

The second author was partially supported by Conacyt (Mexico), Grant # 127794.

equations studied in [23], [7]. We establish the existence and uniqueness of solutions for the Cauchy problem for such equations, see Theorems 4.1, 5.5, 6.3. We show that the fundamental solutions of these equations are transition density functions of Markov processes, see Theorem 7.4. Finally, we study the well-posedness of the Cauchy problem, see Theorem 8.1.

2. PRELIMINARIES

In this section we fix the notation and collect some basic results on p -adic analysis that we will use through the article. For a detailed exposition the reader may consult [1], [19], [22].

2.1. The field of p -adic numbers. Along this article p will denote a prime number. The field of p -adic numbers \mathbb{Q}_p is defined as the completion of the field of rational numbers \mathbb{Q} with respect to the p -adic norm $|\cdot|_p$, which is defined as

$$|x|_p = \begin{cases} 0 & \text{if } x = 0 \\ p^{-\gamma} & \text{if } x = p^\gamma \frac{a}{b}, \end{cases}$$

where a and b are integers coprime with p . The integer $\gamma := \text{ord}(x)$, with $\text{ord}(0) := +\infty$, is called the p -adic order of x . We extend the p -adic norm to \mathbb{Q}_p^n by taking

$$\|x\|_p := \max_{1 \leq i \leq n} |x_i|_p, \quad \text{for } x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n.$$

We define $\text{ord}(x) = \min_{1 \leq i \leq n} \{\text{ord}(x_i)\}$, then $\|x\|_p = p^{-\text{ord}(x)}$. The set $(\mathbb{Q}_p^n, \|\cdot\|_p)$ is a complete ultrametric space. As a topological space \mathbb{Q}_p is homeomorphic to a Cantor-like subset of the real line.

Any p -adic number $x \neq 0$ has a unique expansion $x = p^{\text{ord}(x)} \sum_{j=0}^{\infty} x_j p^j$, where $x_j \in \{0, 1, 2, \dots, p-1\}$ and $x_0 \neq 0$. By using this expansion, we define the *fractional part of $x \in \mathbb{Q}_p$* , denoted $\{x\}_p$, as the rational number

$$\{x\}_p = \begin{cases} 0 & \text{if } x = 0 \text{ or } \text{ord}(x) \geq 0 \\ p^{\text{ord}(x)} \sum_{j=0}^{-\text{ord}(x)-1} x_j p^j & \text{if } \text{ord}(x) < 0. \end{cases}$$

For $\gamma \in \mathbb{Z}$, denote by $B_\gamma^n(a) = \{x \in \mathbb{Q}_p^n : \|x - a\|_p \leq p^\gamma\}$ the ball of radius p^γ with center at $a = (a_1, \dots, a_n) \in \mathbb{Q}_p^n$, and take $B_\gamma^n(0) := B_\gamma^n$. Notice that $B_\gamma^n(a) = B_\gamma(a_1) \times \dots \times B_\gamma(a_n)$, where $B_\gamma(a_i) := \{x \in \mathbb{Q}_p : |x - a_i|_p \leq p^\gamma\}$ is the one-dimensional ball of radius p^γ with center at $a_i \in \mathbb{Q}_p$. The ball B_0^n equals the product of n copies of $B_0 := \mathbb{Z}_p$, the ring of p -adic integers. We denote by $\Omega(\|x\|_p)$ the characteristic function of B_0^n . For more general sets, say Borel sets, we use $1_A(x)$ to denote the characteristic function of A .

2.2. The Bruhat-Schwartz space. A complex-valued function φ defined on \mathbb{Q}_p^n is called *locally constant* if for any $x \in \mathbb{Q}_p^n$ there exists an integer $l = l(x) \in \mathbb{Z}$ such that

$$(2.1) \quad \varphi(x + x') = \varphi(x) \text{ for } x' \in B_l^n.$$

The set of all locally constant functions φ , for which the integer $l(x)$ is independent of x , form \mathbb{C} -vector space denoted by $\tilde{\mathcal{E}}(\mathbb{Q}_p^n) := \tilde{\mathcal{E}}$. Given $\varphi \in \tilde{\mathcal{E}}$, we call the largest possible $l = l(\varphi)$, the *parameter of local constancy of φ* .

A function $\varphi : \mathbb{Q}_p^n \rightarrow \mathbb{C}$ is called a *Bruhat-Schwartz function* (or a *test function*) if it is locally constant with compact support. The \mathbb{C} -vector space of Bruhat-Schwartz functions is denoted by $S(\mathbb{Q}_p^n) := S$. Notice that $S \subset \tilde{\mathcal{E}}$.

Let $S'(\mathbb{Q}_p^n) := S'$ denote the set of all functionals (distributions) on $S(\mathbb{Q}_p^n)$. All functionals on $S(\mathbb{Q}_p^n)$ are continuous.

Set $\chi_p(y) = \exp(2\pi i \{y\}_p)$ for $y \in \mathbb{Q}_p$. The map $\chi_p(\cdot)$ is an additive character on \mathbb{Q}_p , i.e. a continuous map from \mathbb{Q}_p into the unit circle satisfying $\chi_p(y_0 + y_1) = \chi_p(y_0)\chi_p(y_1)$, $y_0, y_1 \in \mathbb{Q}_p$.

Given $\xi = (\xi_1, \dots, \xi_n)$ and $x = (x_1, \dots, x_n) \in \mathbb{Q}_p^n$, we set $\xi \cdot x := \sum_{j=1}^n \xi_j x_j$. The Fourier transform of $\varphi \in S(\mathbb{Q}_p^n)$ is defined as

$$(\mathcal{F}\varphi)(\xi) = \int_{\mathbb{Q}_p^n} \Psi(-\xi \cdot x) \varphi(x) d^n x \quad \text{for } \xi \in \mathbb{Q}_p^n,$$

where $d^n x$ is the Haar measure on \mathbb{Q}_p^n normalized by the condition $\text{vol}(B_0^n) = 1$. The Fourier transform is a linear isomorphism from $S(\mathbb{Q}_p^n)$ onto itself satisfying $(\mathcal{F}(\mathcal{F}\varphi))(\xi) = \varphi(-\xi)$. We will also use the notation $\mathcal{F}_{x \rightarrow \xi} \varphi$ and $\hat{\varphi}$ for the Fourier transform of φ .

2.2.1. Fourier transform. The Fourier transform $\mathcal{F}[T]$ of a distribution $T \in S'(\mathbb{Q}_p^n)$ is defined by

$$(\mathcal{F}[T], \varphi) = (T, \mathcal{F}[\varphi]) \quad \text{for all } \varphi \in S(\mathbb{Q}_p^n).$$

The Fourier transform $f \rightarrow \mathcal{F}[f]$ is a linear isomorphism from $S'(\mathbb{Q}_p^n)$ onto $S'(\mathbb{Q}_p^n)$. Furthermore, $T = \mathcal{F}[\mathcal{F}[T](-\xi)]$.

3. A CLASS OF NON-LOCAL OPERATORS

Denote by \mathfrak{M}_λ , with $\lambda \geq 0$, the \mathbb{C} -vector space of all the functions $\varphi \in \tilde{\mathcal{E}}$ satisfying $|\varphi(x)| \leq C(1 + \|x\|_p^\lambda)$. If the function φ depends also on a parameter t , we shall say that φ belongs to \mathfrak{M}_λ *uniformly with respect to t* , if its constant C and its parameter of local constancy do not depend on t . Notice that, if $0 \leq \lambda_1 \leq \lambda_2$, then $\mathfrak{M}_0 \subseteq \mathfrak{M}_{\lambda_1} \subseteq \mathfrak{M}_{\lambda_2}$, and that $S(\mathbb{Q}_p^n) \subseteq \mathfrak{M}_0$.

Take $\mathbb{R}_+ := \{x \in \mathbb{R}; x \geq 0\}$, and fix a function

$$w_\alpha : \mathbb{Q}_p^n \rightarrow \mathbb{R}_+$$

having the following properties:

- (i) $w_\alpha(y)$ is a radial (i.e. $w_\alpha(y) = w_\alpha(\|y\|_p)$) and continuous function;
- (ii) $w_\alpha(y) = 0$ if and only if $y = 0$;
- (iii) there exist constants $C_0, C_1 > 0$, and $\alpha > n$ such that

$$C_0 \|y\|_p^\alpha \leq w_\alpha(\|y\|_p) \leq C_1 \|y\|_p^\alpha \quad \text{for any } y \in \mathbb{Q}_p^n.$$

Set

$$A_{w_\alpha}(\xi) := \int_{\mathbb{Q}_p^n} \frac{1 - \Psi(-y \cdot \xi)}{w_\alpha(\|y\|_p)} d^n y.$$

In [8], we establish that function A_{w_α} is radial, positive, continuous, $A_{w_\alpha}(0) = 0$, and $A_{w_\alpha}(\xi) = A_{w_\alpha}(\|\xi\|_p) = A_{w_\alpha}(p^{-\text{ord}(\xi)})$ is a decreasing function of $\text{ord}(\xi)$, cf.

[8, Lemma 3.2]. In addition, we introduce the following operator:

$$(3.1) \quad (\mathbf{W}_\alpha \varphi)(x) = \int_{\mathbb{Q}_p^n} \frac{\varphi(x-y) - \varphi(x)}{w_\alpha(\|y\|_p)} d^n y, \quad \varphi \in S(\mathbb{Q}_p^n).$$

Lemma 3.1. *If $\alpha - n > \lambda$, then \mathbf{W}_α can be extended to \mathfrak{M}_λ and formula (3.1) holds. Furthermore, $\mathbf{W}_\alpha : \mathfrak{M}_\lambda \rightarrow \mathfrak{M}_\lambda$.*

Proof. Notice that if $\varphi \in \mathfrak{M}_\lambda$, there exists a constant $l = l(\varphi) \in \mathbb{Z}$, such that

$$(3.2) \quad (\mathbf{W}_\alpha \varphi)(x) = \int_{\|y\|_p \geq p^l} \frac{\varphi(x-y) - \varphi(x)}{w_\alpha(\|y\|_p)} d^n y.$$

We now show that $|(\mathbf{W}_\alpha \varphi)(x)| \leq A(1 + \|x\|_p^\lambda)$. By using that $\varphi \in \mathfrak{M}_\lambda$, and $\alpha > n$,

$$|(\mathbf{W}_\alpha \varphi)(x)| \leq C \int_{\|y\|_p \geq p^l} \frac{(1 + \|x-y\|_p^\lambda)}{\|y\|_p^\alpha} d^n y + C'(1 + \|x\|_p^\lambda).$$

Hence, it is sufficient to show that the above integral can be bounded by $A(1 + \|x\|_p^\lambda)$, for some positive constant A . If $\|x\|_p > \|y\|_p$,

$$\begin{aligned} \int_{\|y\|_p \geq p^l} \frac{(1 + \|x-y\|_p^\lambda)}{\|y\|_p^\alpha} d^n y &\leq (1 + \|x\|_p^\lambda) \int_{\|y\|_p \geq p^l} \frac{1}{\|y\|_p^\alpha} d^n y \\ &= B(1 + \|x\|_p^\lambda), \end{aligned}$$

where B is a positive constant. If $\|x\|_p < \|y\|_p$, by using $\alpha - n > \lambda$,

$$\int_{\|y\|_p \geq p^l} \frac{(1 + \|x-y\|_p^\lambda)}{\|y\|_p^\alpha} d^n y \leq \int_{\|y\|_p \geq p^l} \frac{(1 + \|y\|_p^\lambda)}{\|y\|_p^\alpha} d^n y < \infty.$$

If $\|x\|_p = \|y\|_p \geq p^l$, we take $x = p^L u$, $y = p^L v$, with $\|v\|_p = \|u\|_p = 1$, $L \in \mathbb{Z}$, then

$$\begin{aligned} \int_{\|y\|_p = \|x\|_p} \frac{(1 + \|x-y\|_p^\lambda)}{\|y\|_p^\alpha} d^n y &= p^{-L(n-\alpha)} \int_{\|v\|_p = 1} (1 + p^{-L\lambda} \|u-v\|_p^\lambda) d^n v \\ &\leq A \left(\|x\|_p^{-(\alpha-n)} + \|x\|_p^{-(\alpha-n-\lambda)} \right) \leq A'(p, l, \alpha, n, \lambda), \end{aligned}$$

where A, A' are positive constants.

Finally, by (3.2) $\mathbf{W}_\alpha \varphi$ is locally constant. \square

4. PARABOLIC-TYPE EQUATIONS WITH CONSTANT COEFFICIENTS

Consider the following Cauchy problem:

$$(4.1) \quad \begin{cases} \frac{\partial u}{\partial t}(x, t) - \kappa \cdot (\mathbf{W}_\alpha u)(x, t) = f(x, t), & x \in \mathbb{Q}_p^n, t \in (0, T] \\ u(x, 0) = \varphi(x), \end{cases}$$

where, $\alpha > n$, κ , T are positive constants, $\varphi \in \text{Dom}(\mathbf{W}_\alpha) := \mathfrak{M}_\lambda$, with $\alpha - n > \lambda$, f is continuous in (x, t) and belongs to \mathfrak{M}_λ uniformly with respect to t , and $u : \mathbb{Q}_p^n \times [0, T] \rightarrow \mathbb{C}$ is an unknown function.

We say that $u(x, t)$ is a *solution of (4.1)*, if $u(x, t)$ is continuous in (x, t) , $u(\cdot, t) \in \text{Dom}(W_\alpha)$ for $t \in [0, T]$, $u(x, \cdot)$ is continuously differentiable for $t \in (0, T]$, $u(x, t) \in \mathfrak{M}_\lambda$ uniformly in t , and u satisfies (4.1) for all $t > 0$.

Cauchy problem (4.1) was studied in [8] using semigroup theory. In this article, we study this problem in the space \mathfrak{M}_λ , which is not contained in L^ρ for any $\rho \in [1, \infty]$, and thus we cannot use semigroup theory, see e.g. [8, Theorem 6.5].

We define

$$(4.2) \quad Z(x, t; w_\alpha, \kappa) := Z(x, t) = \int_{\mathbb{Q}_p^n} e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)} \Psi(x \cdot \xi) d^n \xi,$$

for $t > 0$ and $x \in \mathbb{Q}_p^n$. Notice that, $Z(x, t) = \mathcal{F}_{\xi \rightarrow x}^{-1}[e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)}] \in L^1 \cap L^2$ for $t > 0$, since $C' \|\xi\|_p^{\alpha-n} \leq A_{w_\alpha}(\|\xi\|_p) \leq C'' \|\xi\|_p^{\alpha-n}$, cf. [8, Lemma 3.4]. Furthermore, $Z(x, t) \geq 0$, for $t > 0$, $x \in \mathbb{Q}_p^n$, cf. [8, Theorem 4.3 (i)]. These functions are called *heat kernels*. When considering $Z(x, t)$ as a function of x for t fixed we will write $Z_t(x)$.

We set

$$u_1(x, t) := \int_{\mathbb{Q}_p^n} Z(x - y, t) \varphi(y) d^n y,$$

$$u_2(x, t) := \int_0^t \int_{\mathbb{Q}_p^n} Z(x - y, t - \theta) f(y, \theta) d^n y d\theta,$$

for $\varphi, f \in \mathfrak{M}_\lambda$ with $\alpha - n > \lambda$, for $0 \leq t \leq T$, and $x \in \mathbb{Q}_p^n$.

The main result of this section is the following:

Theorem 4.1. *The function*

$$u(x, t) = u_1(x, t) + u_2(x, t)$$

is a solution of Cauchy Problem (4.1).

The proof requires several steps.

4.1. Claim $u(x, t) \in \mathfrak{M}_\lambda$. In order to prove this claim, we need some preliminary results.

Remark 4.2. *The function $Z_t(x)$ is radial since it is the inverse Fourier transform of the radial function $e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)}$. Then $Z_t(x)$ is locally constant in $\mathbb{Q}_p^n \setminus \{0\}$. Furthermore, $Z_t(x + y) = Z_t(x)$ if $\|y\|_p < \|x\|_p$ for any $y \in \mathbb{Q}_p^n$ and $x \in \mathbb{Q}_p^n \setminus \{0\}$, and $t > 0$.*

Lemma 4.3. *There exist positive constants C_1, C_2 such that $Z(x, t)$ satisfies the following conditions:*

- (i) $Z(x, t) \leq C_1 t^{-\frac{n}{\alpha-n}}$, for $t > 0$ and $x \in \mathbb{Q}_p^n$;
- (ii) $Z(x, t) \leq C_2 t \|x\|_p^{-\alpha}$, for $t > 0$ and $x \in \mathbb{Q}_p^n \setminus \{0\}$;
- (iii) $Z(x, t) \leq \max\{2^\alpha C_1, 2^\alpha C_2\} t \left(\|x\|_p + t^{\frac{1}{\alpha-n}} \right)^{-\alpha}$, for $t > 0$ and $x \in \mathbb{Q}_p^n$;
- (iv) $\int_{\mathbb{Q}_p^n} Z(x, t) d^n x = 1$, for $t > 0$.

Proof. (i) By (4.2) and Lemma 3.4 in [8],

$$Z(x, t) \leq \int_{\mathbb{Q}_p^n} e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)} d^n \xi \leq \int_{\mathbb{Q}_p^n} e^{-C_0 t \|\xi\|_p^{\alpha-n}} d^n \xi.$$

Let m be an integer such that $p^{m-1} \leq t^{\frac{1}{\alpha-n}} \leq p^m$, then

$$Z(x, t) \leq \int_{\mathbb{Q}_p^n} e^{-C_0 t \|p^{-(m-1)} \xi\|_p^{\alpha-n}} d^n \xi,$$

now, by changing variables as $z = p^{-(m-1)} \xi$, we have

$$Z(x, t) \leq p^{-(m-1)n} \int_{\mathbb{Q}_p^n} e^{-C_0 t \|z\|_p^{\alpha-n}} d^n z \leq C_1 t^{-\frac{n}{\alpha-n}}.$$

(ii) It follows from [8, Lemma 4.1]. (iii) The results is obtained from the two following inequalities. If $\|x\|_p \geq t^{\frac{1}{\alpha-n}}$, then $\|x\|_p \geq \frac{\|x\|_p}{2} + \frac{t^{\frac{1}{\alpha-n}}}{2}$ and $\|x\|_p^{-\alpha} \leq 2^\alpha \left(\frac{\|x\|_p}{2} + \frac{t^{\frac{1}{\alpha-n}}}{2} \right)^{-\alpha}$, multiplying by $C_2 t$ and using (ii),

$$Z(x, t) \leq 2^\alpha C_2 t \left(\frac{\|x\|_p}{2} + \frac{t^{\frac{1}{\alpha-n}}}{2} \right)^{-\alpha}.$$

If $\|x\|_p \leq t^{\frac{1}{\alpha-n}}$, then $\frac{\|x\|_p}{2} + \frac{t^{\frac{1}{\alpha-n}}}{2} \leq t^{\frac{1}{\alpha-n}}$ and $\left(\frac{\|x\|_p}{2} + \frac{t^{\frac{1}{\alpha-n}}}{2} \right)^{-\alpha} \geq 2^{-\alpha} t^{-\frac{\alpha}{\alpha-n}} = 2^{-\alpha} t^{-1-\frac{n}{\alpha-n}}$, multiplying by C_1 and using (i),

$$Z(x, t) \leq 2^\alpha C_1 t \left(\frac{\|x\|_p}{2} + \frac{t^{\frac{1}{\alpha-n}}}{2} \right)^{-\alpha}.$$

(iv) By (iii), $Z_t(x) \in L^1(\mathbb{Q}_p^n)$ for $t > 0$. Now, the announced identity follows by applying the Fourier inversion formula. \square

Proposition 4.4 ([18], Proposition 2). *If $b > 0$, $0 \leq \lambda < \alpha$, and $x \in \mathbb{Q}_p^n$, then*

$$\int_{\mathbb{Q}_p^n} \left(b + \|x - \xi\|_p \right)^{-\alpha-n} \|\xi\|_p^\lambda d^n \xi \leq C b^{-\alpha} \left(1 + \|x\|_p^\lambda \right),$$

where the constant C does not depend on b or x .

Lemma 4.5. *The functions u_1, u_2 belong to \mathfrak{M}_λ uniformly in t , for $\lambda + n < \alpha$.*

Proof. By Lemma 4.3 (iii), and Proposition 4.4,

$$\begin{aligned} |u_1(x, t)| &\leq \int_{\mathbb{Q}_p^n} Z(x - y, t) |\varphi(y)| d^n y \leq C \int_{\mathbb{Q}_p^n} t \left(t^{\frac{1}{\alpha-n}} + \|x - y\|_p \right)^{-\alpha} \left(1 + \|y\|_p^\lambda \right) d^n y \\ &\leq C' \left(1 + \|x\|_p^\lambda \right). \end{aligned}$$

On the other hand, since

$$u_1(x, t) = \int_{\mathbb{Q}_p^n} Z(w, t) \varphi(x - w) d^n w,$$

u_1 is locally constant and $l(u_1) = l(\varphi)$ uniformly in t . The proof for u_2 is similar. \square

Remark 4.6. Notice that $u_1, u_2, \mathbf{W}_\gamma u_1, \mathbf{W}_\gamma u_2 \in \mathfrak{M}_\lambda$, for any γ satisfying $\lambda + n < \gamma \leq \alpha$.

4.2. Claim $u(x, t)$ satisfies the initial condition. This claim follows from Lemma 4.5 by using the following result.

Lemma 4.7. If $\varphi \in \mathfrak{M}_\lambda$, with $\alpha > \lambda + n$, then

$$\lim_{t \rightarrow 0^+} \int_{\mathbb{Q}_p^n} Z(x - \xi, t) \varphi(\xi) d^n \xi = \varphi(x).$$

Proof. By Lemma 4.3 (iv),

$$(4.3) \quad \int_{\mathbb{Q}_p^n} Z(x - \xi, t) \varphi(\xi) d^n \xi = \int_{\mathbb{Q}_p^n} Z(x - \xi, t) [\varphi(\xi) - \varphi(x)] d^n \xi + \varphi(x).$$

Now, by Lemma 4.3 (iii) and the local constancy of φ ,

$$\begin{aligned} \int_{\mathbb{Q}_p^n} Z(x - \xi, t) [\varphi(\xi) - \varphi(x)] d^n \xi &\leq Ct \int_{\|x - \xi\|_p \geq p^l} (t^{\frac{1}{\alpha-n}} + \|x - \xi\|_p)^{-\alpha} |\varphi(\xi) - \varphi(x)| d^n \xi \\ &\leq Ct \int_{\|z\|_p \geq p^l} (t^{\frac{1}{\alpha-n}} + \|z\|_p)^{-\alpha} |\varphi(x - z) - \varphi(x)| d^n z \\ &\leq Ct \int_{\|z\|_p \geq p^l} \|z\|_p^{-\alpha} (1 + \|x - z\|_p^\lambda) d^n z + C't |\varphi(x)| \leq th(x). \end{aligned}$$

The formula is obtained by taking limit when $t \rightarrow 0^+$ in (4.3). \square

4.3. Claim $u(x, t)$ is a solution of Cauchy problem (4.1). The proof of this claim is a consequence of Corollary 4.10, Lemmas 4.11 and 4.12. Several preliminary results are required.

Lemma 4.8. There exist positive constants C_3, C_4 such that $Z(x, t)$ satisfies the following conditions:

- (i) $\frac{\partial Z(x, t)}{\partial t} = -\kappa \int_{\mathbb{Q}_p^n} A_{w_\alpha}(\|\xi\|_p) e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)} \Psi(x \cdot \xi) d^n \xi$, for $t > 0$ and $x \in \mathbb{Q}_p^n$;
- (ii) $\left| \frac{\partial Z(x, t)}{\partial t} \right| \leq C_3 t^{-\frac{\alpha}{\alpha-n}}$, for $t > 0$ and $x \in \mathbb{Q}_p^n$;
- (iii) $\left| \frac{\partial Z(x, t)}{\partial t} \right| \leq C_4 t \|x\|_p^{n-2\alpha}$, for $t > 0$ and $x \in \mathbb{Q}_p^n \setminus \{0\}$;
- (iv) $\left| \frac{\partial Z(x, t)}{\partial t} \right| \leq 2^\alpha C_3 \left(\|x\|_p + t^{\frac{1}{\alpha-n}} \right)^{-\alpha}$, for $t > 0$ and $x \in \mathbb{Q}_p^n \setminus \{0\}$.

Proof. (i) The formula is obtained by the Lebesgue Dominated Convergence Theorem, and the fact that $-\kappa A_{w_\alpha}(\|\xi\|_p) e^{-\kappa \tau A_{w_\alpha}(\|\xi\|_p)} \Psi(x \cdot \xi) \in L^1(\mathbb{Q}_p^n)$, for $\tau > 0$ fixed, cf. [8, Lemma 3.4]. (ii) By using (i) and Lemma 3.4 in [8],

$$\left| \frac{\partial Z(x, t)}{\partial t} \right| \leq \int_{\mathbb{Q}_p^n} C_1 \|\xi\|_p^{\alpha-n} e^{-\kappa C_2 t \|\xi\|_p^{\alpha-n}} d^n \xi.$$

We now pick an integer m such that $p^{m-1} \leq t^{\frac{1}{\alpha-n}} \leq p^m$, and proceed as in the proof of Lemma 4.3 (i), to obtain

$$\left| \frac{\partial Z(x, t)}{\partial t} \right| \leq C_1 p^{-(m-1)n - (m-1)(\alpha-n)} \int_{\mathbb{Q}_p^n} \|z\|_p^{\alpha-n} e^{-\kappa C_2 t \|z\|_p^{\alpha-n}} d^n z \leq C_3 t^{-\frac{\alpha}{\alpha-n}}.$$

(iii) Set $\|x\|_p = p^\beta$. Now, since $A_{w_\alpha}(\|\xi\|_p) e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)} \in L^1 \cap L^2$ for $t > 0$, then $\frac{\partial Z(x, t)}{\partial t} \in L^1 \cap L^2$ for $t > 0$, and by applying the formula for the Fourier Transform of a radial function, we get

$$\begin{aligned} \frac{\partial Z(x, t)}{\partial t} &= \|x\|_p^{-n} \\ &\times \left((1 - p^{-n}) \sum_{j=0}^{\infty} A_{w_\alpha}(p^{-\beta-j}) e^{-\kappa t A_{w_\alpha}(p^{-\beta-j})} p^{-nj} - A_{w_\alpha}(p^{-\beta+1}) e^{-\kappa t A_{w_\alpha}(p^{-\beta+1})} \right). \end{aligned}$$

Now, by using that $A_{w_\alpha}(\xi)$ is a decreasing function of $\text{ord}(\xi)$,

$$\begin{aligned} \left| \frac{\partial Z(x, t)}{\partial t} \right| &\leq \|x\|_p^{-n} A_{w_\alpha}(p^{-\beta+1}) \left| (1 - p^{-n}) \sum_{j=0}^{\infty} p^{-nj} - e^{-\kappa t A_{w_\alpha}(p^{-\beta+1})} \right| \\ &\leq \|x\|_p^{-n} A_{w_\alpha}(p^{-\beta+1}) \left(1 - e^{-\kappa t A_{w_\alpha}(p^{-\beta+1})} \right) \end{aligned}$$

By using Mean Value Theorem and Lemma 3.4 in [8], we have

$$\left| \frac{\partial Z(x, t)}{\partial t} \right| \leq C_4 \|x\|_p^{n-2\alpha} t.$$

(iv) If $\|x\|_p \leq t^{\frac{1}{\alpha-n}}$, then $\frac{\|x\|_p}{2} + \frac{t^{\frac{1}{\alpha-n}}}{2} \leq t^{\frac{1}{\alpha-n}}$ and $t^{\frac{-\alpha}{\alpha-n}} \leq 2^\alpha \left(\|x\|_p + t^{\frac{1}{\alpha-n}} \right)^{-\alpha}$, multiplying by C_3 and using (ii), we have

$$\left| \frac{\partial Z(x, t)}{\partial t} \right| \leq 2^\alpha C_3 \left(\|x\|_p + t^{\frac{1}{\alpha-n}} \right)^{-\alpha}.$$

Now, if $\|x\|_p \geq t^{\frac{1}{\alpha-n}}$, by using (iii),

$$(4.4) \quad \left| \frac{\partial Z(x, t)}{\partial t} \right| \leq C_3 \|x\|_p^{-\alpha},$$

and since $\|x\|_p \geq t^{\frac{1}{\alpha-n}}$, then $\|x\|_p \geq \left(\frac{\|x\|_p}{2} + \frac{t^{\frac{1}{\alpha-n}}}{2} \right)$ and $2^\alpha \left(\|x\|_p + t^{\frac{1}{\alpha-n}} \right)^{-\alpha} \geq \|x\|_p^{-\alpha}$, multiplying by C_3 and using (4.4), we have

$$\left| \frac{\partial Z(x, t)}{\partial t} \right| \leq 2^\alpha C_3 \left(\|x\|_p + t^{\frac{1}{\alpha-n}} \right)^{-\alpha}.$$

□

Lemma 4.9. $(\mathbf{W}_\gamma Z_t)(x)$, with $\gamma \leq \alpha$, satisfies the following conditions:

- (i) $(\mathbf{W}_\gamma Z_t)(x) = -\int_{\mathbb{Q}_p^n} A_{w_\gamma}(\|\xi\|_p) e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)} \Psi(x \cdot \xi) d^n \xi$, for $t > 0$ and $x \in \mathbb{Q}_p^n$;
- (ii) $|(\mathbf{W}_\gamma Z_t)(x)| \leq 2^\gamma C \left(\|x\|_p + t^{\frac{1}{\alpha-n}} \right)^{-\gamma}$, for $t > 0$ and $x \in \mathbb{Q}_p^n$ and some positive constant C ;
- (iii) $\int_{\mathbb{Q}_p^n} (\mathbf{W}_\gamma Z_t)(x) d^n x = 0$.

Proof. (i) Define

$$(4.5) \quad Z_t^{(M)}(x) = \int_{\|\eta\|_p \leq p^M} \Psi(x \cdot \eta) e^{-\kappa t A_{w_\alpha}(\|\eta\|_p)} d^n \eta, \text{ for } M \in \mathbb{N}.$$

This function is locally constant on \mathbb{Q}_p^n . Indeed, if $\|\xi\|_p \leq p^{-M}$, then $Z_t^{(M)}(x + \xi) = Z_t^{(M)}(x)$. Furthermore, $Z_t^{(M)}(x)$ is bounded, and thus $Z_t^{(M)}(x) \in \mathfrak{M}_0 \subset \text{Dom}(\mathbf{W}_\gamma)$. We now use formula (3.1) and Fubini's Theorem to compute $(\mathbf{W}_\gamma Z_t^{(M)})(x)$ as follows:

$$\begin{aligned} (\mathbf{W}_\gamma Z_t^{(M)})(x) &= \int_{\mathbb{Q}_p^n} \frac{Z_t^{(M)}(x - \xi) - Z_t^{(M)}(x)}{w_\gamma(\|\xi\|_p)} d^n \xi \\ &= \int_{\|\xi\|_p > p^{-M}} \int_{\|\eta\|_p \leq p^M} e^{-\kappa t A_{w_\alpha}(\|\eta\|_p)} \Psi(x \cdot \eta) \frac{(\Psi(\xi \cdot \eta) - 1)}{w_\gamma(\|\xi\|_p)} d^n \eta d^n \xi \\ &= \int_{\|\eta\|_p \leq p^M} e^{-\kappa t A_{w_\alpha}(\|\eta\|_p)} \Psi(x \cdot \eta) \int_{\|\xi\|_p > p^{-M}} \frac{(\Psi(\xi \cdot \eta) - 1)}{w_\gamma(\|\xi\|_p)} d^n \xi d^n \eta \\ &= - \int_{\|\eta\|_p \leq p^M} e^{-\kappa t A_{w_\alpha}(\|\eta\|_p)} \Psi(x \cdot \eta) A_{w_\gamma}(\|\eta\|_p) d^n \eta. \end{aligned}$$

By using that $e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)} A_{w_\gamma}(\|\xi\|_p) \in L^1(\mathbb{Q}_p^n)$ for $t > 0$, cf. [8, Lemma 3.4] and the Dominated Convergence Theorem, we obtain

$$(4.6) \quad \lim_{M \rightarrow \infty} (\mathbf{W}_\gamma Z_t^{(M)})(x) = - \int_{\mathbb{Q}_p^n} A_{w_\gamma}(\|\eta\|_p) e^{-\kappa t A_{w_\alpha}(\|\eta\|_p)} \Psi(x \cdot \eta) d^n \eta.$$

On the other hand, by fixing $x \neq 0$ and for $t > 0$, $Z_t(x - \xi) - Z_t(x)$ is locally constant, cf. Remark 4.2, and bounded, cf. Lemma 4.3 (iii), then $(\mathbf{W}_\gamma Z_t)(x)$ is well-defined, and since $Z_t^{(M)}(x)$ is radial,

$$(\mathbf{W}_\gamma Z_t^{(M)})(x) = \int_{\|\xi\|_p > \|x\|_p} \frac{Z_t^{(M)}(x - \xi) - Z_t^{(M)}(x)}{w_\gamma(\|\xi\|_p)} d^n \xi,$$

and by Dominated Convergence Theorem, $\lim_{M \rightarrow \infty} (\mathbf{W}_\gamma Z_t^{(M)})(x) = (\mathbf{W}_\gamma Z_t)(x)$. Therefore by (4.6), we have

$$(\mathbf{W}_\gamma Z_t)(x) = - \int_{\mathbb{Q}_p^n} A_{w_\gamma}(\|\eta\|_p) e^{-\kappa t A_{w_\alpha}(\|\eta\|_p)} \Psi(x \cdot \eta) d^n \eta.$$

Finally, we note the right-hand side in the above formula is continuous at $x = 0$.

(ii) By (i) and Lemma 3.4 in [8],

$$|(\mathbf{W}_\gamma Z_t)(x)| \leq C_0 \int_{\mathbb{Q}_p^n} \|\xi\|_p^{\gamma-n} e^{-\kappa C_1 t \|\xi\|_p^{\alpha-n}} d^n \xi.$$

We now pick an integer m such that $p^{m-1} \leq t^{\frac{1}{\alpha-n}} \leq p^m$, and proceed as in the proof of Lemma 4.3 (i), to obtain

$$(4.7) \quad |(\mathbf{W}_\gamma Z_t)(x)| \leq Ct^{-\frac{\gamma}{\alpha-n}}.$$

Now, if $\|x\|_p \leq t^{\frac{1}{\alpha-n}}$, then $\frac{\|x\|_p}{2} + \frac{t^{\frac{1}{\alpha-n}}}{2} \leq t^{\frac{1}{\alpha-n}}$ and $t^{\frac{-\gamma}{\alpha-n}} \leq 2^\gamma \left(\|x\|_p + t^{\frac{1}{\alpha-n}} \right)^{-\gamma}$, multiplying by C and by using (4.7), we have

$$|(\mathbf{W}_\gamma Z_t)(x)| \leq 2^\gamma C \left(\|x\|_p + t^{\frac{1}{\alpha-n}} \right)^{-\gamma}.$$

On the other hand, let $\|x\|_p = p^\beta$, since $A_{w_\gamma}(\|\xi\|_p) e^{-\kappa t A_{w_\alpha}(\|\xi\|_p)} \in L^1 \cap L^2$ for $t > 0$, then $(\mathbf{W}_\gamma Z_t)(x) \in L^1 \cap L^2$ for $t > 0$, by proceeding as in the proof of Lemma 4.8 (iii), we obtain

$$|(\mathbf{W}_\gamma Z_t)(x)| \leq Ct \|x\|_p^{n-\alpha-\gamma}.$$

Now, if $\|x\|_p \geq t^{\frac{1}{\alpha-n}}$, then

$$(4.8) \quad |(\mathbf{W}_\gamma Z_t)(x)| \leq C \|x\|_p^{-\gamma}.$$

If $\|x\|_p \geq t^{\frac{1}{\alpha-n}}$, then $\|x\|_p \geq \left(\frac{\|x\|_p}{2} + \frac{t^{\frac{1}{\alpha-n}}}{2} \right)$ and $2^\gamma \left(\|x\|_p + t^{\frac{1}{\alpha-n}} \right)^{-\gamma} \geq \|x\|_p^{-\gamma}$, multiplying by C and using (4.8), we have

$$|(\mathbf{W}_\gamma Z_t)(x)| \leq 2^\gamma C \left(\|x\|_p + t^{\frac{1}{\alpha-n}} \right)^{-\gamma}.$$

(iii) It follows from (i) by the inversion formula for the Fourier transform. \square

Corollary 4.10. $\frac{\partial Z(x,t)}{\partial t} = \kappa \cdot (\mathbf{W}_\alpha Z_t)(x)$ for $t > 0$ and $x \in \mathbb{Q}_p^n$.

Proof. The formula follows from Lemma 4.8 (i) and Lemma 4.9 (i). \square

Proposition 4.11. Assume that $\varphi \in \mathfrak{M}_\lambda$, then the following assertions hold:

- (i) $\frac{\partial u_1}{\partial t}(x, t) = \int_{\mathbb{Q}_p^n} \frac{\partial Z(x-y, t)}{\partial t} \varphi(y) d^n y$, for $t > 0$ and $x \in \mathbb{Q}_p^n \setminus \{0\}$;
- (ii) $(\mathbf{W}_\gamma u_1)(x, t) = \int_{\mathbb{Q}_p^n} (\mathbf{W}_\gamma Z_t)(x-y) \varphi(y) d^n y$, for $n + \lambda < \gamma \leq \alpha$, $t > 0$ and $x \in \mathbb{Q}_p^n \setminus \{0\}$.

Proof. (i) By using the Mean Value Theorem, $\frac{\partial u_1}{\partial t}(x, t)$ equals

$$\lim_{h \rightarrow 0} \int_{\mathbb{Q}_p^n} \left[\frac{Z(x-y, t+h) - Z(x-y, t)}{h} \right] \varphi(y) d^n y = \lim_{h \rightarrow 0} \int_{\mathbb{Q}_p^n} \frac{\partial Z(x-y, \tau)}{\partial t} \varphi(y) d^n y,$$

where τ is between t and $t+h$. Now, the result follows by applying the Dominated Converge Theorem and Lemma 4.8 (iv).

(ii) By Remark 4.6, if $n + \lambda < \gamma$, then $u_1 \in \text{Dom}(\mathbf{W}_\gamma)$ for $t > 0$. Then for any $L \in \mathbb{N}$, the following integral exists:

$$\begin{aligned} & \int_{\|y\|_p > p^{-L}} \frac{u_1(x-y, t) - u_1(x, t)}{w_\gamma(\|y\|_p)} d^n y \\ &= \int_{\|y\|_p > p^{-L}} \frac{1}{w_\gamma(\|y\|_p)} \int_{\mathbb{Q}_p^n} (Z_t(x-y-\xi) - Z_t(x-\xi)) \varphi(\xi) d^n \xi d^n y, \end{aligned}$$

now, by using Fubini's Theorem, cf. Lemma 4.3 (iii),

$$\int_{\mathbb{Q}_p^n} \varphi(\xi) \int_{\|y\|_p > p^{-L}} \frac{(Z_t(x - \xi - y) - Z_t(x - \xi))}{w_\gamma(\|y\|_p)} d^n y d^n \xi.$$

We now fix a positive integer M , such that $\|y\|_p < p^{-L} < p^{-M} < \|x - \xi\|_p$, and use Remark 4.2,

$$\begin{aligned} (\mathbf{W}_\gamma u_1)(x, t) &= \lim_{L \rightarrow \infty} \int_{\|y\|_p > p^{-L}} \frac{u_1(x - y, t) - u_1(x, t)}{w_\gamma(\|y\|_p)} d^n y \\ &= \int_{\|x - \xi\|_p > p^{-M}} \varphi(\xi) (\mathbf{W}_\gamma Z_t)(x - \xi) d^n \xi \\ &\quad + \lim_{L \rightarrow \infty} \int_{\|x - \xi\|_p \leq p^{-M}} \varphi(\xi) \int_{\|y\|_p > p^{-L}} \frac{(Z_t(x - \xi - y) - Z_t(x - \xi))}{w_\gamma(\|y\|_p)} d^n y d^n \xi \\ &= \int_{\|x - \xi\|_p > p^{-M}} \varphi(\xi) (\mathbf{W}_\gamma Z_t)(x - \xi) d^n \xi + \int_{\|x - \xi\|_p \leq p^{-M}} \varphi(\xi) (\mathbf{W}_\gamma Z_t)(x - \xi) d^n \xi. \end{aligned}$$

□

Set

$$u_2(x, t, \tau) := \int_{\tau}^t \int_{\mathbb{Q}_p^n} Z(x - y, t - \theta) f(y, \theta) d^n y d\theta,$$

for $f \in \mathfrak{M}_\lambda$ with $\alpha - n > \lambda$, for $0 \leq \tau \leq t \leq T$, and $x \in \mathbb{Q}_p^n$. By reasoning as in the proof of Lemma 4.5, we have $u_2(x, t, \tau) \in \mathfrak{M}_\lambda$ uniformly in t and τ .

Proposition 4.12. *Assume that $f \in \mathfrak{M}_\lambda$, with $\alpha - n > \lambda$, then the following assertions hold:*

(i) $\frac{\partial u_2}{\partial t}(x, t, \tau) = f(x, t) + \int_{\tau}^t \left(\int_{\mathbb{Q}_p^n} \frac{\partial Z(x - y, t - \theta)}{\partial t} [f(y, \theta) - f(x, \theta)] d^n y \right) d\theta$, for $t > 0$ and $x \in \mathbb{Q}_p^n$;

(ii) $(\mathbf{W}_\gamma u_2)(x, t, \tau) = \int_{\tau}^t \int_{\mathbb{Q}_p^n} (\mathbf{W}_\gamma Z)(x - y, t - \theta) f(y, \theta) d^n y d\theta$, for $n + \lambda < \gamma \leq \alpha$, $t > 0$ and $x \in \mathbb{Q}_p^n$.

Proof. Set

$$u_{2,h}(x, t, \tau) := \int_{\tau}^{t-h} \int_{\mathbb{Q}_p^n} Z(x - y, t - \theta) f(y, \theta) d^n y d\theta, \quad 0 < h < t - \tau.$$

By using a standard reasoning, one shows that

$$\begin{aligned} \frac{\partial u_{2,h}}{\partial t}(x, t, \tau) &= \\ &\int_{\tau}^{t-h} \int_{\mathbb{Q}_p^n} \frac{\partial Z(x - y, t - \theta)}{\partial t} f(y, \theta) d^n y d\theta + \int_{\mathbb{Q}_p^n} Z(x - y, h) f(y, t - h) d^n y d\theta. \end{aligned}$$

This formula can be rewritten as

$$\begin{aligned} \frac{\partial u_{2,h}}{\partial t}(x, t, \tau) &= \int_{\tau}^{t-h} \int_{\mathbb{Q}_p^n} \frac{\partial Z(x-y, t-\theta)}{\partial t} [f(y, \theta) - f(x, \theta)] d^n y d\theta \\ &+ \int_{\tau}^{t-h} f(x, \theta) \int_{\mathbb{Q}_p^n} \frac{\partial Z(x-y, t-\theta)}{\partial t} d^n y d\theta + \int_{\mathbb{Q}_p^n} Z(x-y, h) [f(y, t-h) - f(y, t)] d^n y \\ &+ \int_{\mathbb{Q}_p^n} Z(x-y, h) f(y, t) d^n y. \end{aligned}$$

The first integral contains no singularity at $t = \theta$ due to Lemma 4.8 (iv) and the local constancy of f . By Lemma 4.3 (iv), the second integral is equal to zero. The third integral can be written as a sum of the integrals over $\{y \in \mathbb{Q}_p^n \mid \|x-y\|_p \geq p^M\}$ and the complement of this set, one of these integrals is estimated on the basis of the uniform continuity of f , while the other contains no singularity, see Lemma 4.8 (iv). Finally, the fourth integral tends to $f(x, t)$ as $h \rightarrow 0^+$, cf. Lemma 4.7.

(ii) By Lemma 4.5, $\mathbf{W}_\gamma u_{2,h}$ is well-defined for any γ satisfying $n + \lambda < \gamma \leq \alpha$. Then, for any $L \in \mathbb{N}$, the following integral exists:

$$\begin{aligned} (4.9) \quad & \int_{\|\xi\|_p \geq p^{-L}} \frac{[u_{2,h}(x-\xi, t, \tau) - u_{2,h}(x, t, \tau)]}{w_\gamma(\|\xi\|_p)} d^n \xi \\ &= \int_{\tau}^{t-h} \int_{\mathbb{Q}_p^n} \int_{\|\xi\|_p \geq p^{-L}} \frac{[Z(x-\xi-y, t-\theta) - Z(x-y, t-\theta)]}{w_\gamma(\|\xi\|_p)} f(y, \theta) d^n \xi d^n y d\theta. \end{aligned}$$

On the other hand, by Fubini's Theorem,

$$\begin{aligned} & \int_{\|\xi\|_p \geq p^{-L}} \frac{[Z(x-\xi-y, t-\theta) - Z(x-y, t-\theta)]}{w_\gamma(\|\xi\|_p)} d^n \xi \\ &= \int_{\mathbb{Q}_p^n} \Psi((x-y) \cdot \eta) e^{-\kappa(t-\theta)A_{w_\gamma}(\|\eta\|_p)} P_k(\eta) d^n \eta, \end{aligned}$$

where

$$P_k(\eta) = \int_{\|\xi\|_p \geq p^{-L}} \frac{[\Psi(-\xi \cdot \eta) - 1]}{w_\gamma(\|\xi\|_p)} d^n \xi.$$

A simple calculation shows that

$$|P_k(\eta)| \leq C' \|\eta\|_p^{\gamma-n},$$

and then

$$\int_{\|\xi\|_p \geq p^{-L}} \frac{[Z(x-\xi-y, t-\theta) - Z(x-y, t-\theta)]}{w_\gamma(\|\xi\|_p)} d^n \xi \leq C,$$

where the constant does not depend on $x, t \geq h + \tau, L$.

Now, by expressing the right integral of (4.9) as

$$\begin{aligned} & \int_{\tau}^{t-h} \int_{\|x-\xi\|_p > p^{-M}} \int_{\|\xi\|_p \geq p^{-L}} \frac{[Z(x-\xi-y, t-\theta) - Z(x-y, t-\theta)]}{w_{\gamma}(\|\xi\|_p)} f(y, \theta) d^n \xi d^n y d\theta \\ & + \int_{\tau}^{t-h} \int_{\|x-\xi\|_p \leq p^{-M}} \int_{\|\xi\|_p \geq p^{-L}} \frac{[Z(x-\xi-y, t-\theta) - Z(x-y, t-\theta)]}{w_{\gamma}(\|\xi\|_p)} f(y, \theta) d^n \xi d^n y d\theta \end{aligned}$$

where M is a positive integer, such that $\|\xi\|_p < p^{-L} < p^{-M} < \|x-\xi\|_p$, and using the same reasoning as in the final part of the proof of Proposition 4.11 (ii), we obtain

$$(4.10) \quad (\mathbf{W}_{\gamma} u_{2,h})(x, t) = \int_{\tau}^{t-h} \int_{\mathbb{Q}_p^n} (\mathbf{W}_{\gamma} Z)(x-\xi, t-\theta) f(y, \theta) d^n \xi d\theta.$$

Now, by Lemma 4.9 (ii), the fact that $f \in \mathfrak{M}_{\lambda}$, Proposition 4.4, and the Dominated Convergence Theorem, we can take limit as $h \rightarrow 0^+$, which completes the proof when $\gamma < \alpha$. If $\gamma = \alpha$, formula (4.10) remains valid. By using Lemma 4.9 (iii), formula (4.10) can be rewritten as

$$(\mathbf{W}_{\gamma} u_{2,h})(x, t) = \int_{\tau}^{t-h} \int_{\mathbb{Q}_p^n} (\mathbf{W}_{\gamma} Z)(x-\xi, t-\theta) [f(y, \theta) - f(x, \theta)] d^n \xi d\theta.$$

Now, by using the local constancy of f , we can justify the passage to the limit as $h \rightarrow 0^+$, which completes the proof. \square

Remark 4.13. By Lemma 4.3 (iv) and Lemma 4.8 (i), $\int_{\mathbb{Q}_p^n} \frac{\partial Z(x-y, t-\theta)}{\partial t} d^n y = 0$, then

$$\frac{\partial u_2}{\partial t}(x, t, \tau) = f(x, t) + \int_{\tau}^t \left(\int_{\mathbb{Q}_p^n} \frac{\partial Z(x-y, t-\theta)}{\partial t} f(y, \theta) d^n y \right) d\theta,$$

for $t > 0$ and $x \in \mathbb{Q}_p^n$.

5. PARABOLIC-TYPE EQUATIONS WITH VARIABLE COEFFICIENTS

First, we fix the notation that will be used through this section. We fix $N+1$ positive real numbers satisfying $n < \alpha_1 < \alpha_2 < \dots < \alpha_N < \alpha$. We fix $N+2$ functions $a_k(x, t)$, $k = 0, \dots, N$ and $b(x, t)$ from $\mathbb{Q}_p^n \times [0, T]$ to \mathbb{R} , here T is a positive constant. We assume that: (i) $b(x, t)$ and $a_k(x, t)$, for $k = 0, \dots, N$, belong (with respect to x) to \mathfrak{M}_0 uniformly with respect to $t \in [0, T]$; (ii) $a_0(x, t)$ satisfies the Hölder condition in t with exponent $v \in (0, 1)$ uniformly in x . We also assume the uniform parabolicity condition $a_0(x, t) \geq \mu > 0$ and that $\alpha_{N+1} := n + (\alpha - n)(1 - v) > \alpha_N$. Notice that $\alpha_{N+1} < \alpha$.

Set $\widetilde{\mathbf{W}} := \sum_{k=1}^N a_k(x, t) \mathbf{W}_{\alpha_k} - b(x, t) \mathbf{I}$ with domain \mathfrak{M}_{λ} , and $0 \leq \lambda + n < \alpha_1$. Notice that $\widetilde{\mathbf{W}} : \mathfrak{M}_{\lambda} \rightarrow \mathfrak{M}_{\lambda}$.

In this section we construct a solution for the following initial value problem:

$$(5.1) \quad \begin{cases} \frac{\partial u}{\partial t}(x, t) - a_0(x, t)(\mathbf{W}_\alpha u)(x, t) - (\widetilde{\mathbf{W}}u)(x, t) = f(x, t) \\ u(x, 0) = \varphi(x), \end{cases}$$

where $x \in \mathbb{Q}_p^n$, $t \in (0, T]$, $\varphi(x) \in \mathfrak{M}_\lambda$, $f(x, t) \in \mathfrak{M}_\lambda$ uniformly with respect to t , with $0 \leq \lambda < \alpha_1 - n$, and $f(x, t)$ is continuous in (x, t) (if $a_1(x, t) = \dots = a_N(x, t) \equiv 0$ then we shall assume that $0 \leq \lambda < \alpha - n$).

5.1. Parametrized Cauchy problem. We first study the following Cauchy problem:

$$(5.2) \quad \begin{cases} \frac{\partial u}{\partial t}(x, t) - a_0(y, \theta)(\mathbf{W}_\alpha u)(x, t) = 0, \quad x \in \mathbb{Q}_p^n, t \in (0, T] \\ u(x, 0) = \varphi(x), \end{cases}$$

where $y \in \mathbb{Q}_p^n$, $\theta > 0$ are parameters. By taking $\kappa = a_0(y, \theta) \geq \mu > 0$ and applying the results of Section 4, Cauchy problem (5.2) has a fundamental solution given by

$$Z(x, t; y, \theta, w_\alpha, \kappa) := Z(x, t; y, \theta) = \int_{\mathbb{Q}_p^n} \Psi(x \cdot \xi) e^{-a_0(y, \theta)t A_{w_\alpha}(\|\xi\|_p)} d^n \xi,$$

for $t > 0$ and $x \in \mathbb{Q}_p^n$.

Remark 5.1. All statements from the Lemmas 4.3, 4.8, 4.9 hold for $Z(x, t; y, \theta)$ and the involved constants do not depend of y and θ . Thus, we have the following estimates:

$$(5.3) \quad Z(x, t; y, \theta) \leq C_1 t \left(\|x\|_p + t^{\frac{1}{\alpha-n}} \right)^{-\alpha}, \text{ for } t > 0;$$

$$(5.4) \quad \left| \frac{\partial Z(x, t; y, \theta)}{\partial t} \right| \leq C_2 \left(\|x\|_p + t^{\frac{1}{\alpha-n}} \right)^{-\alpha}, \text{ for } t > 0;$$

$$(5.5) \quad |(\mathbf{W}_\gamma Z)(x, t; y, \theta)| \leq C_3 \left(\|x\|_p + t^{\frac{1}{\alpha-n}} \right)^{-\gamma}, \text{ for } t > 0 \text{ and } \gamma \leq \alpha.$$

And the identities:

$$(5.6) \quad \int_{\mathbb{Q}_p^n} Z(x, t; y, \theta) d^n x = 1, \text{ for } t > 0;$$

$$(5.7) \quad \frac{\partial Z(x, t; y, \theta)}{\partial t} = -a_0(y, \theta) \int_{\mathbb{Q}_p^n} A_{w_\alpha}(\|\xi\|_p) e^{-a_0(y, \theta)t A_{w_\alpha}(\|\xi\|_p)} \Psi(x \cdot \xi) d^n \xi, \text{ for } t > 0;$$

$$(5.8) \quad (\mathbf{W}_\gamma Z)(x, t; y, \theta) = - \int_{\mathbb{Q}_p^n} A_{w_\gamma}(\|\xi\|_p) e^{-a_0(y, \theta)t A_{w_\alpha}(\|\xi\|_p)} \Psi(x \cdot \xi) d^n \xi,$$

for $t > 0$ and $\gamma \leq \alpha$;

$$(5.9) \quad \int_{\mathbb{Q}_p^n} (\mathbf{W}_\gamma Z_t)(x, t; y, \theta) d^n x = 0.$$

Lemma 5.2. *There exists a positive constant C , such that*

$$(5.10) \quad \left| \int_{\mathbb{Q}_p^n} \frac{\partial Z(x - y; t, y, \theta)}{\partial t} d^n y \right| \leq C.$$

Proof. See proof Lemma 4.5 in [13]. \square

5.2. Heat potentials. We define the parameterized heat potentials as follows:

$$u(x, t, \tau) := \int_{\tau}^t \int_{\mathbb{Q}_p^n} Z(x - y, t - \theta; y, \theta) f(y, \theta) d^n y d\theta,$$

where $f \in \mathfrak{M}_\lambda$, $0 \leq \lambda < \alpha - n$, f continuous in (y, θ) . By using the same argument given to prove Lemma 4.5, one proves that $u \in \mathfrak{M}_\lambda$ uniformly in t and τ .

We now calculate the derivative with respect to t and the action of the operator \mathbf{W}_γ on $u(x, t, \tau)$ for $n + \lambda < \gamma \leq \alpha$.

Proposition 5.3. *Assume that $f \in \mathfrak{M}_\lambda$, $0 \leq \lambda < \alpha - n$, f continuous in (y, θ) . Then the following assertions hold:*

- (i) $\frac{\partial u(x, t, \tau)}{\partial t} = f(x, t) + \int_{\tau}^t \int_{\mathbb{Q}_p^n} \frac{\partial Z(x - y, t - \theta; y, \theta)}{\partial t} f(y, \theta) d^n y d\theta;$
- (ii) $(\mathbf{W}_\gamma u)(x, t, \tau) = \int_{\tau}^t \int_{\mathbb{Q}_p^n} (\mathbf{W}_\gamma Z)(x - y, t - \theta; y, \theta) f(y, \theta) d^n y d\theta, \gamma \leq \alpha.$

Proof. It is a simple variation of the proof given for Proposition 4.12. \square

The following technical result will be used later on.

Lemma 5.4 ([13], Lemma 4.6). *Let*

$$J(x, \xi, t, \tau) = \int_{\tau}^t (t - \theta)^{-\rho/\beta} (\theta - \tau)^{-\sigma/\beta} \\ \times \left(\int_{\mathbb{Q}_p^n} \left((t - \theta)^{1/\beta} + \|x - \eta\|_p \right)^{-n-b_1} \left((\theta - \tau)^{1/\beta} + \|\eta - \xi\|_p \right)^{-n-b_2} d^n \eta \right) d\theta,$$

where $x, \xi \in \mathbb{Q}_p^n$, $0 \leq \tau < t$, $b_1, b_2 > 0$, $\rho + b_1 < \beta$, $\sigma + b_2 < \beta$. Then

$$J(x, \xi, t, \tau) \leq \\ C \left\{ B \left(1 - \frac{\rho}{\beta}, 1 - \frac{\sigma + b_2}{\beta} \right) \left((t - \tau)^{1/\beta} + \|x - \xi\|_p \right)^{-n-b_1} (t - \tau)^{-\frac{(\rho + \sigma + b_2 - \beta)}{\beta}} \right. \\ \left. + B \left(1 - \frac{\rho + b_1}{\beta}, 1 - \frac{\sigma}{\beta} \right) \left((t - \tau)^{1/\beta} + \|x - \xi\|_p \right)^{-n-b_2} (t - \tau)^{-\frac{(\rho + \sigma + b_1 - \beta)}{\beta}} \right\},$$

where C is a positive constant depends only on b_1, b_2 and $B(\cdot, \cdot)$ denotes the Archimedean Beta function.

The proof is a simple variation of that given by Kochubei for Lemma 4.6 in [13].

5.3. Construction of a solution.

Theorem 5.5. *The Cauchy problem (5.1), has a solution, which can be represented in the form*

$$(5.11) \quad u(x, t) = \int_0^t \int_{\mathbb{Q}_p^n} \Lambda(x, t, \xi, \tau) f(\xi, \tau) d^n \xi d\tau + \int_{\mathbb{Q}_p^n} \Lambda(x, t, \xi, 0) \varphi(\xi) d^n \xi,$$

where the fundamental solution $\Lambda(x, t, \xi, \tau)$, $x, \xi \in \mathbb{Q}_p^n$, $0 \leq \tau < t \leq T$, has the form

$$(5.12) \quad \Lambda(x, t, \xi, \tau) = Z(x - \xi, t - \tau; \xi, \tau) + \mathcal{W}(x, t, \xi, \tau),$$

with

$$(5.13) \quad |\mathcal{W}(x, t, \xi, \tau)| \leq C \left\{ (t - \tau)^{2 - \frac{\lambda}{\alpha - n}} \left[(t - \tau)^{\frac{1}{\alpha - n}} + \|x - \xi\|_p \right]^{-\alpha} + (t - \tau) \sum_{k=1}^{N+1} \left[(t - \tau)^{\frac{1}{\alpha - n}} + \|x - \xi\|_p \right]^{-\alpha_k} \right\}.$$

Furthermore $Z(x, t; y, \theta)$ satisfies the estimates (5.3), (5.4), (5.5), (5.10).

Proof. We use the usual scheme of Levi's method. Thus, we look for a fundamental solution of (5.1) having form (5.12), with

$$\mathcal{W}(x, t, \xi, \tau) = \int_{\tau}^t \int_{\mathbb{Q}_p^n} Z(x - \eta, t - \theta; \eta, \theta) \Phi(\eta, \theta, \xi, \tau) d^n \eta d\theta.$$

Then, we require that

$$(5.14) \quad \begin{aligned} \frac{\partial \Lambda}{\partial t}(x, t, \xi, \tau) - a_0(x, t)(\mathbf{W}_\alpha \Lambda)(x, t, \xi, \tau) - \sum_{k=1}^N a_k(x, t)(\mathbf{W}_{\alpha_k} \Lambda)(x, t, \xi, \tau) \\ + b(x, t)\Lambda(x, t, \xi, \tau) = 0, \end{aligned}$$

for $x \neq 0$, $t > 0$. Now by using (5.12), (5.7)-(5.8) and Proposition 5.3, we have formally

$$\begin{aligned} & \frac{\partial Z}{\partial t}(x - \xi, t - \tau, \xi, \tau) + \Phi(x, t, \xi, \tau) + \int_{\tau}^t \int_{\mathbb{Q}_p^n} \frac{\partial Z(x - \eta, t - \theta; \eta, \theta)}{\partial t} \Phi(\eta, \theta, \xi, \tau) d^n \eta d\theta \\ & - a_0(x, t) \left\{ (\mathbf{W}_\alpha Z)(x - \xi, t - \tau; \xi, \tau) + \int_{\tau}^t \int_{\mathbb{Q}_p^n} (\mathbf{W}_\alpha Z)(x - \eta, t - \theta; \eta, \theta) \times \right. \\ & \left. \Phi(\eta, \theta, \xi, \tau) d^n \eta d\theta \right\} - \sum_{k=1}^N a_k(x, t) \left\{ (\mathbf{W}_{\alpha_k} Z)(x - \xi, t - \tau; \xi, \tau) \right. \\ & \left. + \int_{\tau}^t \int_{\mathbb{Q}_p^n} (\mathbf{W}_{\alpha_k} Z)(x - \eta, t - \theta; \eta, \theta) \Phi(\eta, \theta, \xi, \tau) d^n \eta d\theta \right\} \\ & + b(x, t) \left\{ Z(x - \xi, t - \tau; \xi, \tau) + \int_{\tau}^t \int_{\mathbb{Q}_p^n} Z(x - \eta, t - \theta; \eta, \theta) \Phi(\eta, \theta, \xi, \tau) d^n \eta d\theta \right\} = 0. \end{aligned}$$

By taking

$$\begin{aligned} R(x, t, \xi, \tau) &:= (a_0(x, t) - a_0(\xi, \tau))(\mathbf{W}_\alpha Z)(x - \xi, t - \tau; \xi, \tau) \\ &+ \sum_{k=1}^N a_k(x, t)(\mathbf{W}_{\alpha_k} Z)(x - \xi, t - \tau; \xi, \tau) - b(x, t)Z(x - \xi, t - \tau; \xi, \tau), \end{aligned}$$

one gets that $\Phi(x, t, \xi, \tau)$ satisfies the integral equation

$$(5.15) \quad \Phi(x, t, \xi, \tau) = R(x, t, \xi, \tau) + \int_{\tau}^t \int_{\mathbb{Q}_p^n} R(x, t, \eta, \theta) \Phi(\eta, \theta, \xi, \tau) d^n \eta d\theta.$$

Now, by using (5.5) and (5.3), we obtain

$$\begin{aligned} |R(x, t, \xi, \tau)| &\leq C_0 \left(|a_0(x, t) - a_0(\xi, \tau)| \left((t - \tau)^{\frac{1}{\alpha - n}} + \|x - \xi\|_p \right)^{-\alpha} \right. \\ &\quad \left. + \sum_{k=1}^N \left((t - \tau)^{\frac{1}{\alpha - n}} + \|x - \xi\|_p \right)^{-\alpha_k} \right. \\ (5.16) \quad &\left. + \left((t - \tau)^{\frac{1}{\alpha - n}} + \|x - \xi\|_p \right)^{-\alpha} (t - \tau) \right). \end{aligned}$$

Claim A.

$$|a_0(x, t) - a_0(\xi, \tau)| \left((t - \tau)^{\frac{1}{\alpha - n}} + \|x - \xi\|_p \right)^{-\alpha} \leq C'_1 \left((t - \tau)^{\frac{1}{\alpha - n}} + \|x - \xi\|_p \right)^{-\alpha_{N+1}},$$

where $\alpha_{N+1} = n + (\alpha - n)(1 - v) > \alpha_N$.

Indeed, by the Hölder condition

$$|a_0(x, t) - a_0(\xi, \tau)| \leq C_1(t - \tau)^v + |a_0(x, \tau) - a_0(\xi, \tau)|.$$

Let $l(a_0)$ be the parameter of local constancy of a_0 . Thus, if $\|x - \xi\|_p \leq p^{l(a_0)}$, then $|a_0(x, t) - a_0(\xi, \tau)| \leq C_1(t - \tau)^v$. In the case $\|x - \xi\|_p \leq p^{l(a_0)}$, the inequality

follows from the fact that $(t-\tau)^v((t-\tau)^{\frac{1}{\alpha-n}} + \|x-\xi\|_p)^{-\alpha+\alpha_{N+1}}$ is bounded, which in turn follows from $\lim_{t \rightarrow \tau} (t-\tau)^{v+\frac{-\alpha+\alpha_{N+1}}{\alpha-n}} = 1$. In the case $\|x-\xi\|_p > p^{l(a_0)}$, taking $|a_0(x, t) - a_0(\xi, \tau)| \leq C_0$, the inequality follows from

$$((t-\tau)^{\frac{1}{\alpha-n}} + \|x-\xi\|_p)^{-\alpha+\alpha_{N+1}} \leq \|x-\xi\|_p^{-\alpha+\alpha_{N+1}} \leq p^{(-\alpha+\alpha_{N+1})l(a_0)}.$$

Claim B.

$$(t-\tau)((t-\tau)^{\frac{1}{\alpha-n}} + \|x-\xi\|_p)^{-\alpha} \leq C_2((t-\tau)^{\frac{1}{\alpha-n}} + \|x-\xi\|_p)^{-\alpha_{N+1}}.$$

This assertion is a consequence of the fact that $\lim_{t \rightarrow \tau} (t-\tau)^{1+\frac{-\alpha+\alpha_{N+1}}{\alpha-n}} = 0$. Now from (5.16), and Claims A-B, we have

$$(5.17) \quad |R(x, t, \xi, \tau)| \leq C \sum_{k=1}^{N+1} \left[(t-\tau)^{\frac{1}{\alpha-n}} + \|x-\xi\|_p \right]^{-\alpha_k}.$$

We solve integral equation (5.15) by the method of successive approximations:

$$(5.18) \quad \Phi(x, t, \xi, \tau) = \sum_{m=1}^{\infty} R_m(x, t, \eta, \theta),$$

where $R_1 \equiv R$ and

$$R_{m+1}(x, t, \xi, \tau) = \int_{\tau}^t \int_{\mathbb{Q}_p^n} R(x, t, \eta, \theta) R_m(\eta, \theta, \xi, \tau) d^n \eta d\theta, \text{ for } m \geq 1.$$

Claim C.

$$|R_{m+1}(x, t, \xi, \tau)| \leq C(2N+2)^m (t-\tau)^{mv} \frac{(\Gamma(v))^{m+1}}{\Gamma((m+1)v)} \times \sum_{j=1}^{N+1} \left[(t-\tau)^{\frac{1}{\alpha-n}} + \|x-\xi\|_p \right]^{-\alpha_j},$$

for $m \geq 0$, where $\Gamma(\cdot)$ denotes the Archimedean Gamma function.

The proof of this assertion will be given later.

It follows from Claim A, by the Stirling formula, that series (5.18) is convergent and that

$$(5.19) \quad |\Phi(x, t, \xi, \tau)| \leq C_0 \sum_{k=1}^{N+1} \left[(t-\tau)^{\frac{1}{\alpha-n}} + \|x-\xi\|_p \right]^{-\alpha_k}.$$

Now (5.13) follows from (5.19) and Lemma 5.4.

Denote by $u_1(x, t)$ and $u_2(x, t)$ the first and second terms in the right hand side of (5.11). Substituting (5.12) into (5.11), we find that

$$u_1(x, t) = \int_0^t \int_{\mathbb{Q}_p^n} Z(x-\xi, t-\tau; \xi, \tau) f(\xi, \tau) d^n \xi d\tau + \int_0^t \int_{\mathbb{Q}_p^n} Z(x-\eta, t-\theta; \eta, \theta) F(\eta, \theta) d^n \eta d\theta,$$

and

$$u_2(x, t) = \int_{\mathbb{Q}_p^n} Z(x-\xi, t; \xi, 0) \varphi(\xi) d^n \xi + \int_0^t \int_{\mathbb{Q}_p^n} Z(x-\eta, t-\theta; \eta, \theta) G(\eta, \theta) d^n \eta d\theta,$$

where

$$(5.20) \quad F(\eta, \theta) = \int_0^\theta \int_{\mathbb{Q}_p^n} \Phi(\eta, \theta, \xi, \tau) f(\xi, \tau) d^n \xi d\tau,$$

$$(5.21) \quad G(\eta, \theta) = \int_{\mathbb{Q}_p^n} \Phi(\eta, \theta, \xi, 0) \varphi(\xi) d^n \xi.$$

Now, by Proposition 4.4 and (5.19), it follows that

$$|F(\eta, \theta)| \leq C_0(1 + \|\eta\|_p^\lambda), \quad |G(\eta, \theta)| \leq C_1(1 + \|\eta\|_p^\lambda),$$

for all $\eta \in \mathbb{Q}_p^n$ and $\theta \in (0, T]$.

Claim D. The functions F and G belong to $\tilde{\mathcal{E}}$, and their parameters of local constancy do not depend on θ .

We first note that by (5.20)-(5.21), it is sufficient to show that $\Phi(\cdot, \theta, \star, \tau)$ is a locally constant function on $(\mathbb{Q}_p^\times)^n \times \mathbb{Q}_p^n$ and that its parameter of local constancy do not depend on θ and τ . Now, by the recursive definition of the function Φ we see that if L is the parameter local constancy for all the functions $a_k(\cdot, t)$, $b(\cdot, t)$, $(\mathbf{W}_{\alpha_k} Z)(\cdot, t - \tau; \star, \tau)$ and $Z(\cdot, t - \tau; \star, \tau)$ on $(\mathbb{Q}_p^\times)^n \times \mathbb{Q}_p^n$, and if $\|\delta\|_p \leq p^{-L}$, we have

$$R(x + \delta, t, \xi + \delta, \tau) = R(x, t, \xi, \tau).$$

Furthermore, we successively obtain

$$\begin{aligned} R_{m+1}(x + \delta, t, \xi + \delta, \tau) &= \int_\tau^t \int_{\mathbb{Q}_p^n} R(x + \delta, t, \eta, \theta) R_m(\eta, \theta, \xi + \delta, \tau) d^n \eta d\theta \\ &= \int_\tau^t \int_{\mathbb{Q}_p^n} R(x + \delta, t, \zeta + \delta, \theta) R_m(\zeta + \delta, \theta, \xi + \delta, \tau) d^n \zeta d\theta \\ &= R_{m+1}(x, t, \xi, \tau), \end{aligned}$$

so that $\Phi(x + \delta, t, \xi + \delta, \tau) = \Phi(x, t, \xi, \tau)$, and hence

$$\begin{aligned} F(\eta + \delta, \theta) &= \int_0^\theta \int_{\mathbb{Q}_p^n} \Phi(\eta + \delta, \theta, \xi, \tau) f(\xi, \tau) d^n \xi d\tau \\ &= \int_0^\theta \int_{\mathbb{Q}_p^n} \Phi(\eta, \theta, \xi + \delta, \tau) f(\xi + \delta, \tau) d^n \xi d\tau = F(\eta, \theta). \end{aligned}$$

Similarly, $G(\eta + \delta, \theta) = G(\eta, \theta)$ when $|\delta|_p \leq p^{-L}$. Thus $u_1(x, t)$, $u_2(x, t) \in \mathfrak{M}_\lambda$ uniformly in t . Thus the potentials in the expressions for $u_1(x, t)$, $u_2(x, t)$ satisfy the conditions to use the differentiation formulas given in Proposition 5.3. By using these formulas along with Proposition 5.3, (5.7)-(5.8) and (5.15), one verifies after simple transformations that $u(x, t)$ is a solution of Cauchy problem (5.1).

Let us show that $u(x, t) \rightarrow \varphi(x)$ as $t \rightarrow 0^+$. Due to (5.12) and (5.13), it is sufficient to verify that

$$v(x, t) := \int_{\mathbb{Q}_p^n} Z(x - \xi, t; \xi, 0) \varphi(\xi) d^n \xi \rightarrow \varphi(x) \text{ as } t \rightarrow 0^+.$$

By virtue of formula (5.6), we have

$$\begin{aligned} v(x, t) &= \int_{\mathbb{Q}_p^n} [Z(x - \xi, t; \xi, 0) - Z(x - \xi, t; x, 0)] \varphi(\xi) d^n \xi \\ &\quad + \int_{\mathbb{Q}_p^n} Z(x - \xi, t; x, 0) [\varphi(\xi) - \varphi(x)] d^n \xi + \varphi(x). \end{aligned}$$

Now, since $Z(x - \xi, t; \cdot, 0)$ and $\varphi(\cdot)$ are locally constant functions, it follows that in both integrals the integration is actually performed over the set

$$\left\{ \xi \in \mathbb{Q}_p^n : \|\xi - x\|_p \geq p^{-L} \right\}.$$

By applying (5.3) on this set, we see that both integrals tend to zero as $t \rightarrow 0^+$.

Proof of Claim C. We use induction on m . The case $m = 0$ is (5.17). We assume the case m , then

$$\begin{aligned} |R_{m+1}(x, t, \xi, \tau)| &\leq \int_{\tau}^t \int_{\mathbb{Q}_p^n} |R(x, t, \eta, \theta)| |R_m(\eta, \theta, \xi, \tau)| d^n \eta d\theta \\ &\leq C_0 (2N + 2)^{m-1} \frac{(\Gamma(v))^m}{\Gamma(mv)} \sum_{j,k=1}^{N+1} \int_{\tau}^t (\theta - \tau)^{(m-1)v} \times \\ &\quad \int_{\mathbb{Q}_p^n} \left[(\theta - \tau)^{\frac{1}{\alpha-n}} + \|\eta - \xi\|_p \right]^{-\alpha_j} \left[(t - \theta)^{\frac{1}{\alpha-n}} + \|x - \eta\|_p \right]^{-\alpha_k} d^n \eta d\theta. \end{aligned}$$

Now by Lemma 5.4, with $-\sigma = (m-1)(\alpha-n)v$, $\rho = 0$, $-n - b_2 = -\alpha_j$, $-n - b_1 = -\alpha_k$, $\beta = \alpha - n$, we have

$$\begin{aligned} &|R_{m+1}(x, t, \xi, \tau)| \\ &\leq C_0 \left\{ (2N + 2)^{m-1} \frac{(\Gamma(v))^m}{\Gamma(mv)} \sum_{j,k=1}^{N+1} B \left(1, \frac{\alpha + (m-1)(\alpha-n)v - \alpha_j}{\alpha - n} \right) \times \right. \\ &\quad \left((t - \tau)^{1/(\alpha-n)} + \|x - \xi\|_p \right)^{-\alpha_k} (t - \tau)^{\frac{(m-1)(\alpha-n)v - \alpha_j + \alpha}{\alpha - n}} \\ &\quad + B \left(\frac{\alpha - \alpha_k}{\alpha - n}, \frac{\alpha - n + (m-1)(\alpha-n)v}{\alpha - n} \right) \left((t - \tau)^{1/(\alpha-n)} + \|x - \xi\|_p \right)^{-\alpha_j} \times \\ &\quad \left. (t - \tau)^{\frac{(m-1)(\alpha-n)v - \alpha_k + \alpha}{\alpha - n}} \right\}. \end{aligned}$$

By using $B(z_1 + \epsilon, z_2 + \delta) \leq B(z_1, z_2)$, for $\epsilon, \delta \geq 0$,

$$B\left(1, \frac{\alpha + (m-1)(\alpha-n)v - \alpha_j}{\alpha-n}\right) \leq B(v, mv),$$

$$B\left(\frac{\alpha - \alpha_k}{\alpha-n}, \frac{\alpha-n + (m-1)(\alpha-n)v}{\alpha-n}\right) \leq B(v, mv),$$

and

$$(t-\tau)^{\frac{(m-1)(\alpha-n)v - \alpha_r + \alpha}{\alpha-n}} = (t-\tau)^{mv - \frac{(\alpha-n)v + \alpha_r - \alpha}{\alpha-n}} \leq C(t-\tau)^{mv},$$

for $1 \leq r \leq N+1$, we get

$$|R_{m+1}(x, t, \xi, \tau)| \leq C(2N+2)^m \frac{(\Gamma(v))^m}{\Gamma((m+1)v)} (t-\tau)^{mv}$$

$$\times \sum_{k=1}^{N+1} \left((t-\tau)^{1/(\alpha-n)} + \|x - \xi\|_p \right)^{-\alpha_k}.$$

□

6. UNIQUENESS OF THE SOLUTION

We recall that $\tilde{\mathcal{E}}$ is the \mathbb{C} -vector space of all functions $\varphi : \mathbb{Q}_p^n \rightarrow \mathbb{C}$, such that there exist a ball B_l^n , with l depending only on φ , and $\varphi(x+x') = \varphi(x)$ for any $x' \in B_l^n$. Notice that $\mathfrak{M}_\lambda \subset \tilde{\mathcal{E}}$ for any λ . We identify any element of $\tilde{\mathcal{E}}$ with a distribution on \mathbb{Q}_p^n . We now recall the following fact: $T \in S'$ with $\text{supp}(T) \subset B_N^n$ if and only if $\hat{T} \in \tilde{\mathcal{E}}$ and its parameter of local constancy is greater than $-N$, cf. [22, pg 109] or [19, Proposition 3.17].

Lemma 6.1. $\mathbf{W}_\alpha : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}$ is a well-defined linear operator. Furthermore,

$$(\mathbf{W}_\alpha \varphi)(x) = -\mathcal{F}_{\xi \rightarrow x}^{-1} \left(A_{w_\alpha}(\|\xi\|_p) \mathcal{F}_{x \rightarrow \xi} \varphi \right).$$

Proof. Let l be a parameter of local constancy of φ , then

$$(\mathbf{W}_\alpha \varphi)(x) = \int_{\|y\|_p \geq p^l} \frac{\varphi(x-y) - \varphi(x)}{w_\alpha(\|y\|_p)} d^n y$$

$$= \frac{1_{\mathbb{Q}_p^n \setminus B_l^n}(x)}{w_\alpha(\|x\|_p)} * \varphi(x) - \varphi(x) \left(\int_{\|y\|_p \geq p^l} \frac{d^n y}{w_\alpha(\|y\|_p)} \right).$$

Then by taking the Fourier transform in S' :

$$\mathcal{F}(\mathbf{W}_\alpha \varphi)(\xi) = \left(\int_{\mathbb{Q}_p^n} 1_{\mathbb{Q}_p^n \setminus B_l^n}(x) \frac{(\Psi(x \cdot \xi) - 1)}{w_\alpha(\|x\|_p)} d^n x \right) (\mathcal{F}\varphi)(\xi),$$

and since $\mathcal{F}\varphi \in S'$ with $\text{supp}(\mathcal{F}\varphi) \subset B_{-l}^n$,

$$\mathcal{F}(\mathbf{W}_\alpha \varphi)(\xi) = \left(\int_{\mathbb{Q}_p^n} \frac{(\Psi(x \cdot \xi) - 1)}{w_\alpha(\|x\|_p)} d^n x \right) \mathcal{F}\varphi(\xi).$$

Therefore,

$$(\mathbf{W}_\alpha \varphi)(x) = -\mathcal{F}_{\xi \rightarrow x}^{-1} \left(A_{w_\alpha}(\|\xi\|_p) \mathcal{F}_{x \rightarrow \xi} \varphi \right) \in \tilde{\mathcal{E}}.$$

□

Take γ be a real number such that $\lambda < \gamma < \alpha_1 - n < \dots < \alpha_N - n < \alpha - n$, and fix a integer L , and set $\psi(x) := p^{Ln} \Omega(p^L \|x\|_p) * \|x\|_p^\gamma$, then

$$\psi(x) = \begin{cases} \|x\|_p^\gamma & \text{if } \|x\|_p > p^{-L} \\ C & \text{if } \|x\|_p \leq p^{-L} \end{cases},$$

and thus $\psi \in \tilde{\mathcal{E}}$.

Lemma 6.2. *With above notation, there exist positive constants C_1 and C_2 such that (i) $|(\mathbf{W}_\alpha \psi)(x)| \leq C_1 \|x\|_p^{\alpha-\gamma+n}$ and (ii) $|(\mathbf{W}_{\alpha_k} \psi)(x)| \leq C_2 \|x\|_p^{\alpha_k-\gamma+n}$, for $k = 1, \dots, N$.*

Proof. By Lemma 6.1,

$$(\mathbf{W}_\alpha \psi)(x) = -\mathcal{F}_{\xi \rightarrow x}^{-1} \left(A_{w_\alpha}(\|\xi\|_p) \Omega(p^{-L} \|\xi\|_p) \frac{\|\xi\|_p^{-\gamma-n}}{\Gamma_n(n+\gamma)} \right) \text{ in } S',$$

where $\Gamma_n(n+\gamma) = \frac{1-p^\gamma}{1-p^{-\gamma-n}}$. Now, since $A_{w_\alpha}(\|\xi\|_p) \Omega(p^{-L} \|\xi\|_p) \frac{\|\xi\|_p^{-\gamma-n}}{\Gamma_n(n+\gamma)}$ is radial and locally integrable, by applying the formula for Fourier transform of radial function,

$$\begin{aligned} (\mathbf{W}_\alpha \varphi)(x) = & \frac{-\|x\|_p^{-n}}{\Gamma_n(n+\gamma)} [(1-p^{-n}) \|x\|_p^{\gamma+n} \sum_{j=0}^{\infty} A_{w_\alpha}(\|x\|_p^{-1} p^{-j}) \Omega(\|x\|_p^{-1-j} p^{-j}) p^{j(\gamma+n)-jn} \\ & - A_{w_\alpha}(\|x\|_p p^{-j}) \Omega(\|x\|_p^{-1} p^{-L+1}) \|x\|_p^{\gamma+n}], \end{aligned}$$

as a distribution on $\mathbb{Q}_p^n \setminus \{0\}$, now by using Lemma 3.4 in [8],

$$|(\mathbf{W}_\alpha \varphi)(x)| \leq C' \left[(1-p^{-n}) \sum_{j=0}^{\infty} p^{-j(\alpha-n)+j\gamma} - p^{(-L+1)(\alpha-n)} \right] \|x\|_p^{-\alpha+n+\gamma}.$$

The proof of (ii) is similar. □

Theorem 6.3. *Assume that the coefficients $a_k(x, t)$, $k = 0, 1, \dots, N$ are non-negative bounded continuous functions, $b(x, t)$ is a bounded continuous function, $0 \leq \lambda < \alpha_1 - n$ (if $a_1(x, t) = \dots = a_k(x, t) \equiv 0$, we shall suppose that $0 \leq \lambda < \alpha - n$) and $u(x, t)$ is a solution of Cauchy problem (5.1) with $f(x, t) = \varphi(x) \equiv 0$ that belongs to class \mathfrak{M}_λ . Then $u(x, t) \equiv 0$.*

Proof. We may assume that $b(x, t) \geq 0$, otherwise we take $u(x, t)e^{\lambda t}$ with $\lambda > b(x, t)$. We prove that $u(x, t) \geq 0$. By contradiction, suppose that $u(x', t') < 0$, for some $x' \in \mathbb{Q}_p^n$ and $t' \in (0, T]$. By Lemma 6.2, it follows that $(\mathbf{W}_\alpha \psi)(x)$ and $(\mathbf{W}_{\alpha_k} \psi)(x) \rightarrow 0$ as $\|x\|_p \rightarrow \infty$, and thus

$$M := \sup_{\substack{0 \leq t \leq T, \\ x \in \mathbb{Q}_p^n}} \left\{ a_0(x, t) |(\mathbf{W}_\alpha \psi)(x)| + \sum_{k=1}^N a_k(x, t) |(\mathbf{W}_{\alpha_k} \psi)(x)| \right\} < \infty.$$

We pick $\rho > 0$ such that $u(x', t') + T\rho < 0$, and then $\sigma > 0$ such that

$$(6.1) \quad u(x', t') + T\rho + \sigma\psi(x') < 0$$

$$(6.2) \quad \rho - \sigma M < 0.$$

We now consider the function

$$v(x, t) := u(x, t) + t\rho + \sigma\psi(x).$$

From (6.1), it follows that $v(x', t') < 0$, so that

$$\inf_{\substack{0 \leq t \leq T, \\ x \in \mathbb{Q}_p^n}} v(x, t) < 0.$$

On the other hand, since $u(x, t) \in \mathfrak{M}_\lambda$, $\lim_{\|x\|_p \rightarrow \infty} \frac{u(x, t)}{\psi(x)} = 0$ and thus $\lim_{\|x\|_p \rightarrow \infty} v(x, t) > 0$ for any $t > 0$. This implies that there exist $x_0 \in \mathbb{Q}_p^n$ and $t_0 \in (0, T]$, such that

$$\inf_{\substack{0 \leq t \leq T, \\ x \in \mathbb{Q}_p^n}} v(x, t) = \min_{\substack{0 \leq t \leq T, \\ x \in \mathbb{Q}_p^n}} v(x, t) = v(x_0, t_0) < 0,$$

and thus, by formula (3.1), $(\mathbf{W}_\alpha v)(x_0, t_0) \geq 0$, $(\mathbf{W}_{\alpha_k} v)(x_0, t_0) \geq 0$ for all k , and $\frac{\partial v}{\partial t}(x_0, t_0) \leq 0$, hence

$$\frac{\partial v}{\partial t}(x_0, t_0) - a_0(x, t)(\mathbf{W}_\alpha v)(x_0, t_0) - \sum_{k=1}^N a_k(x, t)(\mathbf{W}_{\alpha_k} v)(x_0, t_0) + b(x, t)v(x_0, t_0) < 0.$$

Now, by (6.2),

$$\begin{aligned} & \frac{\partial v}{\partial t}(x, t) - a_0(x, t)(\mathbf{W}_\alpha v)(x, t) - \sum_{k=1}^N a_k(x, t)(\mathbf{W}_{\alpha_k} v)(x, t) + b(x, t)v(x, t) \\ &= \rho - \sigma \left[a_0(x, t)(\mathbf{W}_\alpha \psi)(x) + \sum_{k=1}^N a_k(x, t)(\mathbf{W}_{\alpha_k} \psi)(x) \right] + b(x, t)[\rho t + \sigma\psi(x)] \\ & \geq \rho - \sigma M > 0. \end{aligned}$$

We have obtained a contradiction, thus $u(x, t) \geq 0$. Finally taking $-u(x, t)$ instead of $u(x, t)$, we conclude that $u(x, t) \equiv 0$. \square

7. MARKOV PROCESSES

In this section we show that the fundamental solution $\Lambda(x, t, \xi, \tau)$ of Cauchy problem (5.1) is the transition density of a Markov process. We need some preliminary results.

Lemma 7.1. *If the coefficients $a_k(x, t)$ and $b(x, t)$ are nonnegative, then*

$$\Lambda(x, t, \xi, \tau) \geq 0.$$

Proof. It is sufficient to show that

$$u(x, t) = \int_{\mathbb{Q}_p^n} \Lambda(x, t, \xi, \tau) \varphi(\xi) d^m \xi \geq 0,$$

where $u(x, t)$ is the solution of Cauchy problem (5.1) with $f(x, t) \equiv 0$, and initial condition $u(x, 0) = \varphi(x) \geq 0$ with $\varphi \in S(\mathbb{Q}_p^n)$. From (5.12), (5.13), and Lemma 4.3 (iii), it follows that

$$(7.1) \quad u(x, t) \rightarrow 0 \text{ as } \|x\|_p \rightarrow \infty.$$

Now, if $u(x, t) < 0$, then there exist $x_0 \in \mathbb{Q}_p^n$ and $t_0 \in (0, T]$ such that

$$(7.2) \quad \inf_{\substack{0 \leq t \leq T, \\ x \in \mathbb{Q}_p^n}} u(x, t) = u(x_0, t_0) < 0.$$

This implies that $(\mathbf{W}_\alpha u)(x_0, t_0) \geq 0$, $(\mathbf{W}_{\alpha_k} u)(x_0, t_0) \geq 0$ for all k , and $\frac{\partial u}{\partial t}(x_0, t_0) \leq 0$. On the other hand,

$$\frac{\partial u}{\partial t}(x, t) - a_0(x, t)(\mathbf{W}_\alpha u)(x, t) - \sum_{k=1}^N a_k(x, t)(\mathbf{W}_{\alpha_k} u)(x, t) = 0.$$

By using the uniform parabolicity condition $a_0(x, t) \geq \mu > 0$, we get $(\mathbf{W}_\alpha u)(x_0, t_0) = 0$, then by (3.1), $u(x, t_0)$ is constant, and by (7.1), $u(x, t_0) \equiv 0$, which contradicts (7.2). \square

Lemma 7.2. *If $b(x, t) \equiv 0$, then*

$$\int_{\mathbb{Q}_p^n} \Lambda(x, t, \xi, \tau) d^n \xi = 1.$$

Proof. By integrating (5.14) in the variable ξ over whole the space \mathbb{Q}_p^n , and by using Lemma 4.9 (iii), we have

$$\frac{\partial}{\partial t} \left(\int_{\mathbb{Q}_p^n} \Lambda(x, t, \xi, \tau) d^n \xi \right) = 0,$$

thus $\int_{\mathbb{Q}_p^n} \Lambda(x, t, \xi, \tau) d^n \xi$ is independent of t . Now, by integrating (5.12) over whole space \mathbb{Q}_p^n in variable ξ and by using Lemma 4.3 (iv), we have

$$\int_{\mathbb{Q}_p^n} \Lambda(x, t, \xi, \tau) d^n \xi = 1 + \int_{\tau}^t \int_{\mathbb{Q}_p^n} \int_{\mathbb{Q}_p^n} Z(x - \eta, t - \theta, \eta, \theta) \phi(\eta, \theta, \xi, \tau) d^n \eta d^n \xi d\theta.$$

The result is obtained by taking $t = \tau$ in the above formula. \square

Lemma 7.3. *If $b(x, t) \equiv 0$ and $f(x, t) \equiv 0$, then the function $\Lambda(x, t, \xi, \tau)$ satisfies the following property:*

$$(7.3) \quad \Lambda(x, t, \xi, \tau) = \int_{\mathbb{Q}_p^n} \Lambda(x, t, y, \sigma) \Lambda(y, \sigma, \xi, \tau) d^n y.$$

Proof. Consider the following initial value problem:

$$(7.4) \quad \begin{cases} \frac{\partial u}{\partial t}(x, t) - a_0(x, t)(\mathbf{W}_\alpha u)(x, t) - (\widetilde{\mathbf{W}}u)(x, t) = 0 \\ u(x, \tau) = \varphi(x), \quad x \in \mathbb{Q}_p^n \text{ and } t \in (\tau, \sigma], \end{cases}$$

by Theorem 5.5, $u(x, \sigma) = \int_{\mathbb{Q}_p^n} \Lambda(x, \sigma, \xi, \tau) \varphi(\xi) d^n \xi$. Now consider

$$(7.5) \quad \begin{cases} \frac{\partial u}{\partial t}(x, t) - a_0(x, t)(\mathbf{W}_\alpha u)(x, t) - (\widetilde{\mathbf{W}}u)(x, t) = 0 \\ u(x, \sigma) = \int_{\mathbb{Q}_p^n} \Lambda(x, \sigma, \xi, \tau) \varphi(\xi) d^n \xi, \quad x \in \mathbb{Q}_p^n, t \in (\sigma, T], \text{ with } \tau < \sigma < T, \end{cases}$$

by Theorem 5.5 and Fubini's Theorem, the solution of (7.5) is given by

$$u(x, t) = \int_{\mathbb{Q}_p^n} \left(\int_{\mathbb{Q}_p^n} \Lambda(x, t, y, \sigma) \Lambda(y, \sigma, \xi, \tau) d^n y \right) \varphi(\xi) d^n \xi.$$

On the other hand, (7.5) is equivalent to

$$(7.6) \quad \begin{cases} \frac{\partial u}{\partial t}(x, t) - a_0(x, t)(\mathbf{W}_\alpha u)(x, t) - (\widetilde{\mathbf{W}}u)(x, t) = 0 \\ u(x, \tau) = \varphi(x), \quad x \in \mathbb{Q}_p^n, t \in (\tau, T], \end{cases}$$

which has solution given by $u(x, t) = \int_{\mathbb{Q}_p^n} \Lambda(x, t, \xi, \tau) \varphi(\xi) d^n \xi$. Now, by Theorem 6.3,

$$\int_{\mathbb{Q}_p^n} \Lambda(x, t, \xi, \tau) \varphi(\xi) d^n \xi = \int_{\mathbb{Q}_p^n} \left(\int_{\mathbb{Q}_p^n} \Lambda(x, t, y, \sigma) \Lambda(y, \sigma, \xi, \tau) d^n y \right) \varphi(\xi) d^n \xi,$$

for any test function φ , which implies (7.3). \square

Theorem 7.4. *If the coefficients $a_k(x, t)$, $k = 1, \dots, N$ are nonnegative bounded continuous functions, $b(x, t) \equiv 0$, $0 \leq \lambda < \alpha_1 - n$ (if $a_1(x, t) = \dots = a_k(x, t) \equiv 0$, we shall suppose that $0 \leq \lambda < \alpha - n$), and $f(x, t) \equiv 0$, then the fundamental solution $\Lambda(x, t, \xi, \tau)$ is the transition density of a bounded right-continuous Markov process without second kind discontinuities.*

Proof. The result follows from [10, Theorem 3.6] by using Lemmas (7.1)-(7.2)-(7.3), and (5.12)-(5.13), and Lemma 4.3 (iii). \square

8. THE CAUCHY PROBLEM IS WELL-POSED

In this section, we study the continuity of the solution of Cauchy problem (5.1) with respect to $\varphi(x)$ and $f(x, t)$. We assume that the coefficients $a_k(x, t)$, $k = 0, 1, \dots, N$ are nonnegative bounded continuous functions, $b(x, t)$ is a bounded continuous function, $0 \leq \lambda < \alpha_1 - n$ (if $a_1(x, t) = \dots = a_k(x, t) \equiv 0$, we shall suppose that $0 \leq \lambda < \alpha - n$), $\varphi(x) \in \mathfrak{M}_\lambda$ and $f(x, t) \in \mathfrak{M}_\lambda$, uniformly in t , with $0 \leq \lambda < \alpha_1 - n$.

We identify \mathfrak{M}_λ with the \mathbb{R} -vector space of all the functions " $\phi(x, t) \in \mathfrak{M}_\lambda$, uniformly in t ," and introduce on \mathfrak{M}_λ the following norm:

$$\|\phi\|_{\mathfrak{M}_\lambda} := \sup_{t \in [0, T]} \sup_{x \in \mathbb{Q}_p^n} \left| \frac{\phi(x, t)}{1 + \|x\|_p^\lambda} \right|.$$

From now on, we consider \mathfrak{M}_λ as topological vector space with the topology induced by $\|\cdot\|_{\mathfrak{M}_\lambda}$. We also consider $\mathfrak{M}_\lambda \times \mathfrak{M}_\lambda$ as topological vector space with the topology induced by the norm $\|\cdot\|_{\mathfrak{M}_\lambda} + \|\star\|_{\mathfrak{M}_\lambda}$.

Theorem 8.1. *With the above hypotheses, consider the following operator:*

$$\mathfrak{M}_\lambda \times \mathfrak{M}_\lambda \xrightarrow{\mathbf{L}} \mathfrak{M}_\lambda$$

$$(\varphi(x), f(x, t)) \rightarrow u(x, t),$$

where $u(x, t)$ is given by (5.11). Then $\|u(x, t)\|_{\mathfrak{M}_\lambda} \leq C(\|\varphi(x)\|_{\mathfrak{M}_\lambda} + \|f(x, t)\|_{\mathfrak{M}_\lambda})$, i.e. \mathbf{L} is a continuous operator.

Proof. We write $u(x, t) = u_1(x, t) + u_2(x, t)$ where

$$u_1(x, t) = \int_0^t \int_{\mathbb{Q}_p^n} \Lambda(x, t, \xi, \tau) f(\xi, \tau) d^n \xi d\tau \quad \text{and} \quad u_2(x, t) = \int_{\mathbb{Q}_p^n} \Lambda(x, t, \xi, 0) \varphi(\xi) d^n \xi$$

as before. Now

$$\begin{aligned} |u_1(x, t)| &\leq \int_0^t \int_{\mathbb{Q}_p^n} |\Lambda(x, t, \xi, \tau)| |f(\xi, \tau)| d^n \xi d\tau \\ &\leq \|f(x, t)\|_{\mathfrak{M}_\lambda} \left\{ \int_0^t \int_{\mathbb{Q}_p^n} |\Lambda(x, t, \xi, \tau)| d^n \xi d\tau + \int_0^t \int_{\mathbb{Q}_p^n} |\Lambda(x, t, \xi, \tau)| \|\xi\|_p^\lambda d^n \xi d\tau \right\}, \end{aligned}$$

by (5.12)-(5.13), (5.3) and Proposition 4.4,

$$\begin{aligned} |u_1(x, t)| &\leq C_0 \|f(x, t)\|_{\mathfrak{M}_\lambda} \left\{ \int_0^t (t - \tau)^{1 + \frac{n - \alpha}{\alpha - n}} d\tau + \int_0^t (t - \tau)^{2 - \frac{\lambda}{\alpha - n} + \frac{n - \alpha}{\alpha - n}} d\tau \right. \\ &\quad + \sum_{k=1}^{N+1} \int_0^t (t - \tau)^{1 + \frac{n - \alpha k}{\alpha - n}} d\tau + \left(1 + \|x\|_p^\lambda\right) \int_0^t (t - \tau)^{1 + \frac{n - \alpha}{\alpha - n}} d\tau \\ &\quad + \left(1 + \|x\|_p^\lambda\right) \int_0^t (t - \tau)^{2 - \frac{\lambda}{\alpha - n} + \frac{n - \alpha}{\alpha - n}} d\tau + \left(1 + \|x\|_p^\lambda\right) \times \\ &\quad \left. \sum_{k=1}^{N+1} \int_0^t (t - \tau)^{1 + \frac{n - \alpha k}{\alpha - n}} d\tau \right\} \leq \|f(x, t)\|_{\mathfrak{M}_\lambda} \left\{ C_1(T) + C_2(T) \left(1 + \|x\|_p^\lambda\right) \right\}. \end{aligned}$$

Hence,

$$\left| \frac{u_1(x, t)}{1 + \|x\|_p^\lambda} \right| \leq \|f(x, t)\|_{\mathfrak{M}_\lambda} \left\{ \frac{C_1(T)}{1 + \|x\|_p^\lambda} + C_2(T) \right\}.$$

In the same form, one shows that

$$\left| \frac{u_2(x, t)}{1 + \|x\|_p^\lambda} \right| \leq \|\varphi(x)\|_{\mathfrak{M}_\lambda} \left\{ \frac{C'_1(T)}{1 + \|x\|_p^\lambda} + C'_2(T) \right\},$$

therefore $\|u(x, t)\|_{\mathfrak{M}_\lambda} \leq C(\|\varphi(x, t)\|_{\mathfrak{M}_\lambda} + \|f(x, t)\|_{\mathfrak{M}_\lambda})$. \square

Acknowledgement 8.2. *The authors wish to thank to Sergii Torba for many useful comments and discussions, which led to an improvement of this work.*

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